Setup

Start with the standard basis:

e1 = (1,0), e2 = (0,1)

The Walsh–Hadamard transform sends them to:

$$e_1 \mapsto \frac{1}{\sqrt{2}}(1,1), \quad e_2 \mapsto \frac{1}{\sqrt{2}}(1,-1)$$

We'll now reproduce exactly this mapping in three different ways.

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1) Reflection in the 22.5° line

A reflection across a line at angle Θ has matrix:

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$$R_{\text{ref}}(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$R_{\text{ref}}(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Set $\theta = 22.5^\circ$, so $2\theta = 45^\circ$, and:

$$\cos 45^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

So the matrix becomes exactly H.

Apply to the points

- For $e_1 = (1,0)$:

$$\mapsto \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

- For $e_2 = (0,1)$:

$$\mapsto \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

✔ Matches Hadamard exactly.

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2) Reflection in x-axis, then rotation by 45°

Step 1: reflect in x-axis

$$(x, y) \mapsto (x, -y)$$

- $e_1 = (1,0) \to (1,0)$
- $e_2 = (0,1) \to (0,-1)$

Step 2: rotate by 45°

Rotation matrix:

$$R(45^\circ) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Apply it:

- For transformed $e_1 = (1, 0)$:

$$\mapsto \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

- For transformed $e_2 = (0, -1)$:

$$\mapsto \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

✔ Same result again.

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3) Eigenbasis (Hadamard basis) viewpoint

Define new basis:

$$v_+ = \frac{1}{\sqrt{2}}(1, 1), \quad v_- = \frac{1}{\sqrt{2}}(1, -1)$$

Step 1: express inputs in this basis

- $e_1 = (1, 0)$:

$$= \frac{1}{\sqrt{2}} v_+ + \frac{1}{\sqrt{2}} v_-$$

- $e_2 = (0, 1)$:

$$= \frac{1}{\sqrt{2}} v_+ - \frac{1}{\sqrt{2}} v_-$$

In this basis:

- $v_+ \to$ unchanged
- $v_- \to$ sign flip

So:

- For e_1:

$$\frac{1}{\sqrt{2}}v_+ + \frac{1}{\sqrt{2}}v_- \mapsto \frac{1}{\sqrt{2}}v_+ - \frac{1}{\sqrt{2}}v_-$$

- For e_2:

$$\frac{1}{\sqrt{2}}v_+ - \frac{1}{\sqrt{2}}v_- \mapsto \frac{1}{\sqrt{2}}v_+ + \frac{1}{\sqrt{2}}v_-$$

Convert back to standard coordinates:

- For e_1:

$$= (1/\sqrt{2})(1,1)$$

- For e_2:

$$= (1/\sqrt{2})(1,-1)$$

✔ Same result again.

Final synthesis

All three constructions produce identical mappings:

Method	Interpretation	Mechanism

22.5° line	Single reflection	Global geometric mirror
x-reflection + 45° rotation	Composition	Simple operations in sequence
Eigenbasis	Diagonal form	Keep one component, flip the other

Key insight

What's really happening is:

- The vectors (1,1) and (1,-1) define the **natural symmetry axes**
- In that coordinate system, the transform is trivial (just a sign flip)
- In standard coordinates, that trivial operation *appears* as a reflection across a tilted line

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If you extend this exact "track the basis vectors through simple operations" approach to 4-point Hadamard, you'll see the recursive butterfly structure emerge almost mechanically.